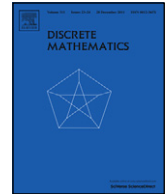




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## Vertex coloring without large polychromatic stars

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### ABSTRACT

Given an integer  $k \geq 2$ , we consider vertex colorings of graphs in which no  $k$ -star subgraph  $S_k = K_{1,k}$  is polychromatic. Equivalently, in a  $star$ - $[k]$ -coloring the closed neighborhood  $N[v]$  of each vertex  $v$  can have at most  $k$  different colors on its vertices. The maximum number of colors that can be used in a  $star$ - $[k]$ -coloring of graph  $G$  is denoted by  $\tilde{\chi}_{k*}(G)$  and is termed the  $star$ - $[k]$  upper chromatic number of  $G$ .

We establish some lower and upper bounds on  $\tilde{\chi}_{k*}(G)$ , and prove an analogue of the Nordhaus–Gaddum theorem. Moreover, a constant upper bound (depending only on  $k$ ) can be given for  $\tilde{\chi}_{k*}(G)$ , provided that the complement  $\bar{G}$  admits a  $star$ - $[k]$ -coloring with more than  $k$  colors.

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### 1. Introduction

In the bulk of graph coloring theory, the main issue is to use a fairly small number of colors, whilst the local constraint forces different colors for adjacent vertices. In the present note we study a kind of coloring where we restrict the possible number of colors locally to be small and ask how rich overall colorings it allows on the global level. Making another comparison, similarly to achromatic number, here the total number of colors is to be maximized; but instead of each color seeing all the others, just to the opposite we aim at minimizing interactions between color classes.

More formally, for a given integer  $k \geq 2$ , a  $star$ - $[k]$ -coloring (shortly introduced under the name ‘ $[k]$ -coloring’ in [5]) is a coloring of the vertices such that the closed neighborhood of each vertex contains vertices of at most  $k$  different colors. This means that no  $k$ -star  $S_k = K_{1,k}$  can be totally multicolored. Or equivalently, each vertex  $v$  is allowed to be adjacent to vertices of at most  $k - 1$  color classes different from that of  $v$ . The  $star$ - $[k]$  upper chromatic number  $\tilde{\chi}_{k*}(G)$  of  $G$  is the maximum number of colors that can occur in such a coloring.

**Remark 1.** (i) According to our definition we will assume throughout that  $k$  is at least 2. But it is worth noting that for  $k = 1$ , the above definition would also make sense: it would yield a coloring in which the vertex set in each connected component is monochromatic; and then  $\tilde{\chi}_{1*}(G)$  would simply be equal to the number of components of  $G$ .

(ii) In hypergraph terminology,  $C$ -coloring means that each hyperedge has at least two vertices assigned to the same color (‘ $C$ ’ originating from ‘common’ color), and the maximum number of colors in such a coloring is termed the upper chromatic number,  $\bar{\chi}(\mathcal{H})$ , of hypergraph  $\mathcal{H}$ . When applied to graph  $G$ , this yields that no edge can have its two ends properly 2-colored, therefore the entire vertex set of any component is monochromatic and  $\tilde{\chi}_{1*}(G) = \bar{\chi}(G)$  would hold for  $k = 1$ . For this kind of color assignment, however, we reserve the name ‘ $C$ -coloring’, and all theorems throughout this paper are meant under the assumption  $k \geq 2$ .

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(iii) In general, for  $k \geq 2$ ,  $\bar{\chi}_{k*}(G)$  is equal to the upper chromatic number of the hypergraph whose hyperedges are the vertex sets of the subgraphs isomorphic to the  $k$ -star  $S_k = K_{1,k}$ , and a vertex coloring of this hypergraph is a C-coloring if and only if it is a star- $[k]$ -coloring of  $G$ .

(iv) Alternatively,  $\bar{\chi}_{k*}(G)$  can be defined as the maximum order of a partition of the vertex set  $V(G)$ , such that for every vertex  $v$  in  $G$  the closed neighborhood  $N[v]$  is contained in the union of at most  $k$  partition classes.

As a motivation of studying star- $[k]$ -coloring, we can consider the following example. In a network of some locations exactly one facility is available at each location. Residents of each location  $v$  are entitled to use any different facility which is available at an adjacent location or in  $v$ , with the restriction that they can use at most  $k$  different kinds of facilities. Direct links between locations mean that residents should be able to use the facilities of their neighbors. The maximum number of facilities that can be made available at different locations in the network satisfying the above condition is equal to the star- $[k]$  upper chromatic number of the network.

Putting it in the other way round, assume that the users of a network would like to access at least a given number of resources altogether. The question is, how high we should raise local accessibility/compatibility to other kinds of resources in order to reach this goal.

For graph theory terminology not defined here we refer to [3].

### 1.1. Some simple facts

It is worth taking some simple observations already at this early point. They follow directly from the definition of star- $[k]$ -coloring.

If  $k \leq k'$ , then every star- $[k]$ -coloring is a star- $[k']$ -coloring, as well. Hence, we get:

**Observation 2.** For any graph  $G$  and for any two positive integers  $k \leq k'$ ,  $\bar{\chi}_{k*}(G) \leq \bar{\chi}_{k'*}(G)$  holds.

**Observation 3.** The star- $[k]$  upper chromatic number is additive for a vertex-disjoint union. That is, if  $G_1$  and  $G_2$  have no common vertex, then for their union  $\bar{\chi}_{k*}(G_1 \cup G_2) = \bar{\chi}_{k*}(G_1) + \bar{\chi}_{k*}(G_2)$  holds.

If  $H$  is a spanning subgraph of  $G$ , then  $H$  can be obtained by deleting some edges of  $G$ . Hence any star- $[k]$ -coloring of  $G$  can be regarded as a star- $[k]$ -coloring of  $H$ , as well.

**Observation 4.** If  $H$  is a spanning subgraph of  $G$ , then  $\bar{\chi}_{k*}(G) \leq \bar{\chi}_{k*}(H)$  holds.

Consider a star- $[k]$ -coloring  $\varphi$  of  $G$  which uses the colors  $c_1, \dots, c_{\ell-1}, c_\ell$ . If every occurrence of color  $c_\ell$  is replaced with color  $c_{\ell-1}$ , the coloring constraint remains fulfilled. Starting with a star- $[k]$ -coloring which uses  $\bar{\chi}_{k*}(G)$  colors, and repeatedly applying the above recoloring, we obtain the following observation which has already been stated in [5]:

**Observation 5.** For every graph  $G$  and for every integer  $\ell$  between 1 and  $\bar{\chi}_{k*}(G)$ ,  $G$  admits a star- $[k]$ -coloring with precisely  $\ell$  colors.

### 1.2. Related coloring concepts

The star- $[k]$ -coloring can be regarded as a generalization of the 3-consecutive C-coloring. For a given graph  $G = (V, E)$ , a mapping  $\varphi : V \rightarrow \mathbb{N}$  is a 3-consecutive C-coloring if there exists no 3-colored path  $P_3 \cong S_2$ ; that is, the closed neighborhood  $N[v]$  of each vertex  $v$  is allowed to contain vertices of at most two different colors [2]. The 3-consecutive upper chromatic number  $\bar{\chi}_{3CC}(G)$  of  $G$  is the maximum number of colors that can be used in such a coloring.

When  $k = 2$ , we find that  $\bar{\chi}_{2*}(G)$  is the 3-consecutive upper chromatic number  $\bar{\chi}_{3CC}(G)$  of  $G$  and hence, due to Observation 2, we have:

**Proposition 6.** For every graph  $G$ , and every  $k \geq 2$ ,  $\bar{\chi}_{3CC}(G) \leq \bar{\chi}_{k*}(G)$  holds.

Another related notion introduced previously is the following one. For an integer  $k \geq 1$ ,  $k$ -improper C-coloring of a graph  $G = (V, E)$  is defined as a vertex coloring  $\varphi : V \rightarrow \mathbb{N}$  such that for every vertex  $v \in V$  at most  $k$  vertices in the neighborhood  $N(v)$  of  $v$  receive colors different from that of  $v$  [1]. The  $k$ -improper upper chromatic number  $\bar{\chi}_{k\text{-imp}}(G)$  is the maximum number of colors that can be used in such a coloring of  $G$ .

Note that in a  $k$ -improper C-coloring we prescribe an upper bound on the number of neighboring vertices having different colors from  $v$ ; whilst in a star- $[k]$ -coloring we bound the number of different color classes in which the closed neighborhood  $N[v]$  of  $v$  has some vertices. Clearly, any  $(k-1)$ -improper C-coloring is a star- $[k]$ -coloring, hence we have:

**Proposition 7.** For every graph  $G$ ,  $\bar{\chi}_{(k-1)\text{-imp}}(G) \leq \bar{\chi}_{k*}(G)$  holds.

### 1.3. Exact values of $\bar{\chi}_{k*}$ for some graphs

The following values of  $\bar{\chi}_{k*}$  either can be checked directly or can be deduced from the results of later sections of this paper.

- For the path  $P_n$  on  $n$  vertices,  $\bar{\chi}_{k*}(P_n) = \begin{cases} \lfloor n/2 \rfloor + 1 & \text{if } k = 2, \\ n & \text{if } k \geq 3. \end{cases}$
- For the cycle  $C_n$  on  $n \geq 4$  vertices,  $\bar{\chi}_{k*}(C_n) = \begin{cases} \lfloor n/2 \rfloor & \text{if } k = 2, \\ n & \text{if } k \geq 3. \end{cases}$
- For the wheel  $W_n$  on  $n + 1$  vertices,  $\bar{\chi}_{k*}(W_n) = \min(n + 1, k)$ .
- For the complete graph  $K_n$ ,  $\bar{\chi}_{k*}(K_n) = \min(n, k)$ .
- If  $G$  is the Petersen graph, then  $\bar{\chi}_{k*}(G) = \begin{cases} 2 & \text{if } k = 2, \\ 5 & \text{if } k = 3, \\ 10 & \text{if } k \geq 4. \end{cases}$

## 2. Extremal values

In this section we deal with graphs having the possible largest or smallest  $\bar{\chi}_{k*}$ .

**Proposition 8.** For any graph  $G$  of order  $p$  and with maximum degree  $\Delta(G)$ ,  $\bar{\chi}_{k*}(G) = p$  holds if and only if  $\Delta(G) \leq k - 1$ .

**Proof.** Sufficiency is obvious. Hence, suppose that  $\bar{\chi}_{k*}(G) = p$  and  $\Delta(G) \geq k$ . Choose a vertex  $v$  with degree  $\Delta(G)$ . In a star- $[k]$ -coloring there can occur at most  $k$  different colors on the  $\Delta(G) + 1 \geq k + 1$  vertices of the closed neighborhood  $N[v]$ . This would imply  $\bar{\chi}_{k*}(G) < p$ , which is a contradiction.  $\square$

In view of Proposition 8, henceforth we concentrate on cases where  $k \leq \Delta(G)$ .

Since an arbitrary distribution of at most  $k$  colors always obeys the requirement of star- $[k]$ -coloring, we have:

**Proposition 9.** For any graph  $G$  of order  $p \geq k$ ,  $\bar{\chi}_{k*}(G) \geq k$ .

Next, we take some simple observations on graphs which have no star- $[k]$ -coloring with more than  $k$  colors. A vertex adjacent to all the other vertices in a graph  $G$  is called a *universal vertex*. The following fact follows by definition.

**Proposition 10.** If  $G$  is a graph of order  $p \geq k$  with at least one universal vertex, then  $\bar{\chi}_{k*}(G) = k$ .

Hence, graphs of radius 1 have the smallest possible star- $[k]$  upper chromatic number. But if the diameter is at least 3, this is surely not the case.

**Proposition 11.** For every graph  $G$  of order greater than  $k$ , if  $\text{diam}(G) \geq 3$ , then  $\bar{\chi}_{k*}(G) > k$ .

**Proof.** Given a graph  $G = (V, E)$  satisfying the condition, let  $x$  and  $y$  be two vertices at a distance at least 3 apart. Consider the following vertex coloring  $\varphi$ :  $\varphi(x) = 1$ ,  $\varphi(y) = 2$ , and the remaining at least  $k - 1$  vertices are arbitrarily assigned with further exactly  $k - 1$  colors. By the choice of  $x$  and  $y$ , they do not belong to a common star subgraph in  $G$ . Consequently, no star is colored with  $k + 1$  different colors, and hence  $\varphi$  is a star- $[k]$ -coloring with exactly  $k + 1$  colors.  $\square$

The situation is not so simple for graphs of radius 2 and diameter 2. Some of them, e.g. the complete bipartite graph  $K_{m,n}$  with  $m \geq k - 1 \geq 2$  and  $n \geq 2$  can be colored with more than  $k$  colors, using exactly  $k - 1$  colors in the first vertex class and further  $\min(k - 1, n)$  colors in the second class. On the other hand, e.g. for the complete graph minus a perfect matching,  $\bar{\chi}_{k*}(K_{2s} - sK_2) = k$  holds if  $2s \geq k + 2$ . Indeed, assume for a contradiction a star- $[k]$ -coloring  $\varphi$  with  $k + 1$  colors and let  $c$  be a color occurring on more than one vertex. If  $\varphi(x) = c$  and  $xy$  is the edge omitted from  $x$ , the closed neighborhood  $N[y] = V \setminus \{x\}$  of  $y$  would contain all the  $k + 1$  colors, contradicting the coloring constraint.

So, the following problem remains open.

**Problem 12.** Characterize the graphs  $G$  for which  $\bar{\chi}_{k*}(G) = k$ .

If a tree is of radius 2 then its diameter is either 3 or 4, hence the problematic case does not arise. The trees of radius 1 are exactly the stars.

**Proposition 13.** For a tree  $T$  of order greater than  $k$ ,  $\bar{\chi}_{k*}(T) = k$  holds if and only if  $T$  is a star.

More generally in bipartite graphs, arbitrarily distributing at most  $k - 1$  distinct colors in each vertex class we can never violate the conditions of star- $[k]$ -coloring. Hence, the following lower bound holds.

**Proposition 14.** For any bipartite graph  $G$  with  $m$  and  $n$  vertices in its partite sets,  $\bar{\chi}_{k*}(G) \geq \min(m, k - 1) + \min(n, k - 1)$ .

**Problem 15.** Characterize the bipartite graphs  $G \subseteq K_{m,n}$  for which the equality  $\bar{\chi}_{k*}(G) = \min(m, k - 1) + \min(n, k - 1)$  is valid.

## 3. Relating $\bar{\chi}_{k*}$ to other invariants

We begin with an upper bound in terms of maximum degree.

**Proposition 16.** If  $G$  is a graph of order  $p$  and maximum degree  $\Delta$ , then  $\bar{\chi}_{k*}(G) \leq p - \Delta + k - 1$  holds.

**Proof.** If  $v$  is a vertex of degree  $\Delta$ , then in any star- $[k]$ -coloring of  $G$ , at most  $k$  colors occur on the vertices in  $N[v]$ . To color the remaining  $p - (\Delta + 1)$  vertices, we cannot use more than  $p - (\Delta + 1)$  colors. Thus,  $\bar{\chi}_{k*}(G) \leq k + p - (\Delta + 1) = p - \Delta + k - 1$ .  $\square$

It is well known that the chromatic number satisfies the inequality  $\chi(G) \leq \Delta + 1$ . Hence, a relation between  $\chi(G)$  and  $\bar{\chi}_{k*}(G)$  is as follows.

**Proposition 17.** For every graph  $G$ ,  $\bar{\chi}_{k*}(G) \leq p - \chi(G) + k$ .

In terms of minimum degree, the following bound can be proved, which is tight for unions of complete graphs of properly chosen orders.

**Proposition 18.** If  $G$  is of order  $p$  and minimum degree  $\delta$ , then

$$\bar{\chi}_{k*}(G) \leq \left\lfloor \frac{pk}{\delta + 1} \right\rfloor.$$

**Proof.** Let  $\varphi$  be a star- $[k]$ -coloring of  $G$ . Each vertex  $v$  has degree at least  $\delta$ , therefore each color occurs in at least  $\delta + 1$  closed neighborhoods  $N[u]$ . On the other hand, each of the  $p$  closed neighborhoods  $N[u]$  can contain at most  $k$  colors. These estimates put lower and upper bounds on the pairs  $(u, c)$  such that  $c \in \varphi(N[u])$ . Therefore,

$$(\delta + 1) \cdot \bar{\chi}_{k*}(G) \leq pk$$

holds, from which the statement follows.  $\square$

In a graph  $G = (V, E)$  a set  $S \subseteq V$  is a *neighborhood set* of  $G$  if  $\bigcup_{v \in S} \langle N[v] \rangle = G$ , where  $\langle N(v) \rangle$  is the subgraph induced by the closed neighborhood  $N[v]$  of  $v$ . The *neighborhood number* of  $G$ , denoted by  $n_0(G)$ , is the minimum cardinality of a neighborhood set in  $G$  (see [4]).

An upper bound involving  $n_0(G)$  is as follows.

**Theorem 19.** For every connected graph  $G$ ,  $\bar{\chi}_{k*}(G) \leq (k - 1)n_0(G) + 1$  holds.

**Proof.** Let us choose a neighborhood set  $S = \{v_1, \dots, v_m\}$  of minimum cardinality  $m = n_0(G)$ . We index the elements of  $S$  in such a way that  $N[v_i]$  intersects  $\bigcup_{1 \leq j < i} N[v_j]$  for all  $2 \leq i \leq m$ . Since  $G$  is connected, such an order exists. In a star- $[k]$ -coloring each  $N[v_i]$  can involve at most  $k$  colors. Moreover, by the chosen connected indexing,  $\bigcup_{1 \leq j < i} N[v_j]$  already contains at least one color occurring in  $N[v_i]$ . Hence, the total number of colors in a star- $[k]$ -coloring of  $G$  cannot exceed  $(k - 1)m + 1$ .  $\square$

The vertex covering number  $\alpha_0(G)$  of a graph  $G = (V, E)$  is the minimum cardinality of a vertex set  $S \subseteq V$  such that  $S$  contains at least one end of every edge  $e \in E$ . It was observed in [4] that  $n_0(G) \leq \alpha_0(G)$ . Therefore, we have:

**Proposition 20.** For a connected graph  $G$ ,  $\bar{\chi}_{k*}(G) \leq (k - 1)\alpha_0(G) + 1$ .

In a connected graph  $G = (V, E)$ , a set  $D \subseteq V$  is a *connected dominating set* if every vertex in  $V \setminus D$  is adjacent to some vertex in  $D$ , moreover  $D$  induces a connected subgraph in  $G$ . The *connected domination number*  $\gamma_c(G)$  of  $G$  is the minimum cardinality of a connected dominating set [6].

The proof of the following statement is very similar to that of Theorem 19 and hence it is omitted here.

**Theorem 21.** For any connected graph  $G$ ,  $\bar{\chi}_{k*}(G) \leq (k - 1)\gamma_c(G) + 1$ .

#### 4. Nordhaus–Gaddum-type results

The well-known theorem of Nordhaus and Gaddum states that  $2\sqrt{p} \leq \chi(G) + \chi(\bar{G}) \leq p + 1$  is valid for every  $G$  of order  $p$ . Combining this with Proposition 17 for  $G$  and its complement, a rough upper bound  $\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq 2(p - \sqrt{p} + k)$  is obtained.

In this section we improve the upper bound on  $\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G})$  in two ways, one of them being valid for all graphs and another sharp one for graphs of sufficiently large order. Furthermore, we point out that  $\bar{\chi}_{k*}(\bar{G}) > k$  implies a constant upper bound for the star- $[k]$  upper chromatic number of  $G$ , independently of the order of the graph.

First, we establish a general upper bound. Later we will see that it can be tight only for sufficiently small graphs. But this simple observation has the advantage of being universal, it holds for all possible values of  $k$  and  $p$ .

**Proposition 22.** For every graph  $G$  of order  $p$  and for its complement  $\bar{G}$ , the inequality  $\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq p + 2k - 1$  holds, and in case of equality  $G$  must be regular of degree between  $k - 1$  and  $p - k$ .

**Proof.** We are going to give two alternative arguments verifying the upper bound, because they imply together the necessary condition stated in the theorem for the case of equality.

First, choose an arbitrary vertex  $v$  in  $G$ . Its open neighborhoods in  $G$  and in  $\bar{G}$  will be denoted by  $N_1$  and  $N_2$ , respectively. Clearly,  $|N_1| + |N_2| = p - 1$ . In any star- $[k]$ -coloring  $\varphi$  of  $G$ , the closed neighborhood  $N_1 \cup \{v\}$  of  $v$  can be colored with at most  $\min(k, |N_1| + 1)$  colors. Together with the remaining  $|N_2|$  vertices, this means that at most  $|N_2| + \min(k, |N_1| + 1)$  colors can be used in  $\varphi$ . A similar argument proves that  $\bar{G}$  cannot be colored with more than  $|N_1| + \min(k, |N_2| + 1)$  colors. These observations yield

$$\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq |N_1| + |N_2| + \min(k, |N_1| + 1) + \min(k, |N_2| + 1) \leq p + 2k - 1.$$

Equality implies  $|N_1| \geq k - 1$  and  $|N_2| \geq k - 1$ . That is, in case of equality no  $v$  can have a degree smaller than  $k - 1$ , neither in  $G$  nor in its complement.

Alternatively, let us observe that  $\Delta(\bar{G}) = p - 1 - \delta(G)$  holds. Thus, from Proposition 16 we obtain

$$\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq (p - \Delta + k - 1) + (p - (p - 1 - \delta) + k - 1) = p + 2k - 1 + \delta - \Delta \leq p + 2k - 1.$$

This also implies  $\Delta = \delta$  whenever the assertion holds with equality.  $\square$

If at least one of  $G$  and  $\bar{G}$  has star- $[k]$  upper chromatic number  $k$ , the upper bound  $p + 2k - 1$  cannot be tight. In the other case, as we will see later, Theorems 23 and 24 together imply that  $\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq (k^2 - 1) + (3k - 2)$ . Hence, if  $\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) = p + 2k - 1$  holds, then  $p \leq k^2 + k - 2$  is also true. Taking into account also Proposition 22, we obtain that the bound  $p + 2k - 1$  can be attained only if  $2k - 1 \leq p \leq k^2 + k - 2$  and only if  $G$  is regular and its vertices have degree at least  $k - 1$  in both  $G$  and its complement.

The main result of this section is the following constant upper bound on  $\bar{\chi}_{k*}(G)$ .

**Theorem 23.** For every graph  $G$  and for its complement  $\bar{G}$ , the inequality  $\bar{\chi}_{k*}(\bar{G}) \geq k + 1$  implies the upper bound  $\bar{\chi}_{k*}(G) \leq k^2 - 1$ .

**Proof.** From the condition  $\bar{\chi}_{k*}(\bar{G}) \geq k + 1$ , the existence of a star- $[k]$ -coloring  $\bar{\varphi}$  of  $\bar{G} = (V, E')$  with exactly  $k + 1$  colors follows. Its color classes will be denoted by  $C_0, C_1, \dots, C_k$ . We will also consider star- $[k]$ -colorings of graph  $G$  and prove that  $|\varphi(V)| \leq k^2 - 1$  holds for every such coloring  $\varphi$ . We say that a vertex  $v$  and a vertex set  $S$  are totally adjacent (totally nonadjacent) if  $v$  is adjacent (nonadjacent) to every vertex of  $S$ . Similarly, two vertex sets  $S_1$  and  $S_2$  are called totally adjacent (totally nonadjacent) if every vertex  $v \in S_1$  is totally adjacent (totally nonadjacent) to  $S_2$ . By the coloring constraint,

( $\star$ ) For every vertex  $v \in V$ , there exists a color class  $C_i$  such that in  $\bar{G}$ ,  $v$  is totally nonadjacent to  $C_i$  and  $v \notin C_i$ . Equivalently, in  $G$ ,  $v$  is totally adjacent to  $C_i$ .

We distinguish between two cases related to  $\bar{\varphi}$ . First, assume that there exists a ‘universal color’, say  $c_0$ , which occurs in every closed neighborhood  $N_{\bar{G}}[v]$  ( $v \in V$ ). Changing the color of a vertex to this universal color  $c_0$  cannot increase the number of colors in any closed neighborhoods  $N_{\bar{G}}[v]$ . Thus, it can be assumed that the color classes different from  $C_0$  are singletons  $C_i = \{u_i\}$  for every  $1 \leq i \leq k$ .

Define the maximal subsets  $B_i$  of the universal color class for which  $B_i$  and  $u_i$  are totally nonadjacent in  $\bar{G}$ :

$$B_i = \{v \in C_0 \mid vu_i \notin E'\} \quad \text{for } i = 1, \dots, k.$$

Due to ( $\star$ ), every vertex of  $C_0$  belongs to at least one  $B_i$ ; that is,  $\bigcup_{i=1}^k B_i = C_0$ . Moreover, by definition,  $u_i$  and  $B_i$  are totally adjacent in  $G$ . Hence,  $B_i \cup \{u_i\} \subseteq N_G[u_i]$  which allows the occurrence of at most  $k$  different colors on  $B_i \cup \{u_i\}$  in every star- $[k]$ -coloring  $\varphi$  of  $G$ . Further, we can take into account that by ( $\star$ ), every  $u_i$  is nonadjacent to at least one  $u_j$  in  $\bar{G}$  (where  $i \neq j$ ). So, also  $u_j$  belongs to the neighborhood of  $u_i$  in  $G$ . Consequently, if already  $B_i \cup \{u_i\}$  contains  $k$  colors in  $\varphi$ , then at least one of those colors is repeated in another set  $B_j \cup \{u_j\}$ . If there exist exactly  $\ell \geq 0$  sets  $B_i \cup \{u_i\}$  with at most  $k - 1$  colors in  $\varphi$ , then the number  $|\varphi(V)|$  of colors is not greater than

$$\ell(k - 1) + (k - \ell)k - \left\lceil \frac{k - \ell}{2} \right\rceil = k^2 - \left\lceil \frac{k + \ell}{2} \right\rceil \leq k^2 - \lceil k/2 \rceil \leq k^2 - 1.$$

Thus, if a universal color exists in a  $(k + 1)$ -coloring of  $\bar{G}$ , the upper bound  $\bar{\chi}_{k*}(G) \leq k^2 - 1$  is valid.

In the other case we have a star- $[k]$ -coloring  $\bar{\varphi}$  of  $\bar{G}$  with  $k + 1$  colors without a universal color. Then, for every color class  $C_i$  (where  $0 \leq i \leq k$ ), there exists a vertex  $v_i \notin C_i$ , such that  $v_i$  and  $C_i$  are totally nonadjacent in  $\bar{G}$  and totally adjacent in  $G$ . Hence,  $C_i \cup \{v_i\} \subseteq N_G[v_i]$ . This relation and ( $\star$ ) together imply the following facts for every star- $[k]$ -coloring  $\varphi$  of  $G$ :

- $|\varphi(C_i)| \leq k$  holds for every  $0 \leq i \leq k$ .
- For every vertex  $v \in V$ 
  - (a) either  $v$  is totally adjacent to a class  $C_i$  satisfying  $|\varphi(C_i)| \leq k - 1$ ,
  - (b) or  $v$  is totally adjacent to a  $C_i$  with  $|\varphi(C_i)| = k$  colors and in this case the color  $\varphi(v)$  is necessarily repeated in  $C_i$ , outside the color class  $C_j \ni v$ .

A color class of  $\varphi$  is called of type (A) if it is entirely contained in a class  $C_i$ . Each vertex of a type (A) color class satisfies the property (a). All the remaining non-(A)-type colors of  $\varphi$  appear in at least two of the sets  $C_i$ .

Let  $\ell$  denote the number of color classes  $C_i$  colored with at most  $k - 1$  colors in  $\varphi$ . By the coloring constraint on  $\varphi$ , each of these  $\ell$  classes can be totally adjacent in  $G$  to vertices from at most  $k - 1$  color classes of type (A). Hence, the number of type (A) classes is not greater than  $\ell(k - 1)$ . All the remaining classes of  $\varphi$  meet at least two sets  $C_i$ . Thus, the following upper bound is valid:

$$|\varphi(V)| \leq \ell(k - 1) + \frac{(k + 1 - \ell)k}{2} = \frac{k^2 + k + \ell(k - 2)}{2}.$$

Since  $k \geq 2$  and  $\ell \leq k + 1$ , we obtain that

$$|\varphi(V)| \leq \frac{k^2 + k + (k + 1)(k - 2)}{2} = k^2 - 1.$$

Consequently, the upper bound  $\bar{\chi}_{k*}(G) \leq k^2 - 1$  is valid also for this second case, where  $\bar{\varphi}$  has no universal color.  $\square$

The following construction shows that the upper bound  $k^2 - 1$  is sharp for every odd  $k \geq 3$  and for every  $p \geq k^2 - 1$ .

**Construction 1.** Given an odd integer  $k \geq 3$  and an integer  $p \geq k^2 - 1$ , let the  $p$ -element vertex set of  $G$  be partitioned into  $k + 1$  disjoint sets  $A_1, \dots, A_{k+1}$ , such that  $|A_i| \geq k - 1$  for every  $1 \leq i \leq k + 1$ . In  $G$ , let the set-pairs  $(A_{2j-1}, A_{2j})$  be totally adjacent (that is,  $G$  contains the corresponding complete bipartite graphs) for every  $1 \leq j \leq (k + 1)/2$ . The edge set consists only of these edges.

One can see that  $\bar{\chi}_{k*}(\bar{G}) \geq k + 1$ , since the color classes  $A_1, \dots, A_{k+1}$  determine a star- $[k]$ -coloring of  $\bar{G}$ . On the other hand, we can consider the coloring  $\varphi$  of  $G$  which assigns the  $k - 1$  colors  $(i - 1)(k - 1) + 1, \dots, i(k - 1)$  to the vertices of  $A_i$ , for each  $1 \leq i \leq k + 1$ . In view of Theorem 23 and since  $\varphi$  is a star- $[k]$ -coloring of  $G$  with exactly  $k^2 - 1$  colors,  $\bar{\chi}_{k*}(G) = k^2 - 1$  holds.  $\square$

For  $k$  even, we can prove a slightly weaker lower bound,  $k^2 - k/2$  only. Nevertheless, at least its leading term is tight.

**Construction 2.** Given an even integer  $k \geq 2$  and any  $p \geq k^2$ , let the vertex set be the disjoint union of  $2k$  sets. For every  $1 \leq i \leq k$ ,  $A_i$  contains precisely  $k - 1$  vertices, and the remaining  $p - k^2 + k \geq k$  vertices are distributed among the sets  $B_i$  ( $1 \leq i \leq k$ ) in any way such that none of them is empty.

In  $G$ , let the following set-pairs be totally adjacent:  $(A_i, B_i)$  for every  $1 \leq i \leq k$ , and  $(B_{2s-1}, B_{2s})$  for every  $1 \leq s \leq k/2$ . The edge set consists only of these edges.

One can see that  $\bar{G}$  has a star- $[k]$ -coloring with the  $k + 1$  color classes  $B_1, B_2, \dots, B_k$ , and  $A = \bigcup_{i=1}^k A_i$ . On the other hand, we can color  $G$  as follows: each of the  $k^2 - k$  vertices of  $A$  has its dedicated color, and further color classes are  $B_{2s-1} \cup B_{2s}$  for every  $1 \leq s \leq k/2$ . This way, we obtain a star- $[k]$ -coloring of  $G$  with exactly  $k^2 - k/2$  colors.  $\square$

Complementing Theorem 23, we also prove that  $\bar{\chi}_{k*}$  cannot be a superlinear function of  $k$  in both  $G$  and  $\bar{G}$  simultaneously.

**Theorem 24.** For every integer  $k \geq 2$  and for every graph  $G$ ,

$$\min(\bar{\chi}_{k*}(G), \bar{\chi}_{k*}(\bar{G})) \leq 3k - 2.$$

**Proof.** Suppose for a contradiction that  $G$  is a counterexample. Let  $\varphi$  and  $\bar{\varphi}$  be star- $[k]$ -colorings of  $G$  and  $\bar{G}$ , respectively, with  $3k - 1$  colors each.

Let us fix an arbitrary vertex  $v_0$ , and choose vertex subsets  $S, S'$  (not necessarily disjoint) such that  $|S| = |S'| = 3k - 2$ , moreover  $S \cup \{v_0\}$  is totally multicolored in  $\varphi$  and similarly  $S' \cup \{v_0\}$  in  $\bar{\varphi}$ . We define

$$S_0 = S \cap N_{\bar{G}}[v_0], \quad S'_0 = S' \cap N_G[v_0].$$

By definition,  $S_0 \cap S'_0 = \emptyset$ . We claim that  $|S_0| \geq 2k - 1$  and  $|S'_0| \geq 2k - 1$ . Indeed, since  $\varphi$  is a star- $[k]$ -coloring and  $S \cup \{v_0\}$  is multicolored in  $\varphi$ , at most  $k - 1$  neighbors of  $v_0$  can belong to  $S$  in  $G$ . Hence  $|S_0| \geq |S| - k + 1 = 2k - 1$ . The lower bound on  $|S'_0|$  follows in the same way.

Consider now the pairs  $vv'$  where  $v \in S_0$  and  $v' \in S'_0$ . We may assume that at least half of them are edges in  $G$ . (For  $\bar{G}$  the argument is analogous.) It means that the edge density in the bipartite subgraph of  $G$  with vertex classes  $S_0$  and  $S'_0$  is at least  $1/2$ . Thus, there exists a vertex  $v' \in S'_0$  having at least  $\lceil |S_0|/2 \rceil \geq k$  neighbors in  $S_0$ . By definition,  $v'$  is also adjacent to  $v_0$ . Moreover,  $S_0 \cup \{v_0\}$  is multicolored, therefore at least  $k + 1$  colors appear in the closed neighborhood of  $v'$ . This contradicts the assumption that  $\varphi$  is a star- $[k]$ -coloring.  $\square$

We do not know at present how tight the upper bound  $3k - 2$  is. But, at least,  $2k - 1$  is a lower bound, as shown by the following example.

**Construction 3.** Let  $p \geq 4k - 4 \geq 4$  and split the vertex set into four parts,  $V = V_1 \cup V_2 \cup V_3 \cup V_4$  where  $|V_i| \geq k - 1$  for  $i = 1, \dots, 4$ . We completely join  $V_i$  with  $V_{i+1}$  for  $i = 1, 2, 3$  and insert edges inside each  $V_i$  arbitrarily.

This graph  $G$  has  $\bar{\chi}_{k*}(G) = 2k - 1$ ; a star- $[k]$ -coloring is obtained by assigning the colors  $1, \dots, k - 1$  in an arbitrary distribution to the vertices of  $V_1$ , colors  $k, \dots, 2k - 2$  to  $V_4$ , and color  $2k - 1$  to the entire  $V_2 \cup V_3$ . More colors are not possible (the set  $V_1 \cup V_3$  can contain at most  $k - 1$  colors different from that of any  $v \in V_2$ , and the same upper bound is valid for  $V_2 \cup V_4$ ). Since  $\bar{G}$  has the same structure, we also have  $\bar{\chi}_{k*}(\bar{G}) = 2k - 1$ .  $\square$



**Problem 25.** Given  $k \geq 1$ , determine  $\max_G \{\min(\bar{\chi}_{k*}(G), \bar{\chi}_{k*}(\bar{G}))\}$ .

As an easy consequence of [Theorems 23](#) and [24](#), the analog of the Nordhaus–Gaddum theorem can be obtained for graphs of sufficiently large order.

**Theorem 26.** For every integer  $k \geq 2$  and for every graph  $G$  of order  $p \geq k^2 + 2k - 3$ ,

$$\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq p + k$$

holds.

**Proof.** If  $\bar{\chi}_{k*}(\bar{G}) = k$  or  $\bar{\chi}_{k*}(G) = k$ , the upper bound clearly is valid. In the other case we have  $\bar{\chi}_{k*}(\bar{G}) \geq k + 1$  and  $\bar{\chi}_{k*}(G) \geq k + 1$ , moreover

$$\min(\bar{\chi}_{k*}(G), \bar{\chi}_{k*}(\bar{G})) \leq 3k - 2$$

by [Theorem 24](#) and

$$\max(\bar{\chi}_{k*}(G), \bar{\chi}_{k*}(\bar{G})) \leq k^2 - 1$$

by [Theorem 23](#). These yield the inequality

$$\bar{\chi}_{k*}(G) + \bar{\chi}_{k*}(\bar{G}) \leq k^2 + 3k - 3.$$

Since our condition  $k^2 + 2k - 3 \leq p$  guarantees that  $k^2 + 3k - 3 \leq p + k$ , the statement follows.  $\square$

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